

NORM INFLATION FOR GENERALIZED NAVIER-STOKES EQUATIONS

ALEXEY CHESKIDOV AND MIMI DAI

ABSTRACT. We consider the incompressible Navier-Stokes equation with a fractional power $\alpha \in [1, \infty)$ of Laplacian in the three dimensional case. We prove the existence of a smooth solution with arbitrary small in $\dot{B}_{\infty, \infty}^{-\alpha}$ initial data that becomes arbitrary large in $\dot{B}_{\infty, \infty}^{-s}$ for all $s > 0$ in arbitrary small time. This extends the result of Bourgain and Pavlović [1] for the classical Navier-Stokes equation which utilizes the fact that the energy transfer to low modes increases norms with negative smoothness indexes. It is remarkable that the space $\dot{B}_{\infty, \infty}^{-\alpha}$ is supercritical for $\alpha > 1$. Moreover, the norm inflation occurs even in the case $\alpha \geq 5/4$ where the global regularity is known.

KEY WORDS: fractional Navier-Stokes equation; norm inflation; Besov spaces; interactions of plane waves

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1. INTRODUCTION

In this paper we study the three dimensional incompressible Navier-Stokes equations with a fractional power of the Laplacian:

$$(1.1) \quad \begin{aligned} u_t + (u \cdot \nabla)u + \nabla p &= -\nu(-\Delta)^\alpha u, \\ \nabla \cdot u &= 0, \\ u(x, 0) &= u_0, \end{aligned}$$

where $x \in \mathbb{R}^3$, $t \geq 0$, u is the fluid velocity, p is the pressure of the fluid and $\nu > 0$ is the kinematic viscosity coefficient. The initial data u_0 is divergence free. The power $\alpha = 1$ corresponds to the classical Navier-Stokes equations. A vast amount of literature has been devoted to these equations, for background we refer the readers to [5] and [12].

Solutions to the fractional Navier-Stokes equation (1.1) have the following scaling property. If $(u(x, t), p(x, t))$ solves system (1.1) with the initial data $u_0(x)$, then

$$u_\lambda(x, t) = \lambda^{2\alpha-1}u(\lambda x, \lambda^{2\alpha}t), \quad p_\lambda(x, t) = \lambda^{2(2\alpha-1)}p(\lambda x, \lambda^{2\alpha}t)$$

solves the system (1.1) with the initial data

$$u_{0\lambda} = \lambda^{2\alpha-1}u_0(\lambda x).$$

A space that is invariant under the above scaling is called a critical space. The largest critical space in three dimension for the fractional NSE (1.1) is the Besov space $\dot{B}_{\infty, \infty}^{1-2\alpha}$ (see [2]).

The study of the Navier-Stokes equations in critical spaces has been one of the focuses of the research activities since the initial work of Kato [6]. In 2001, Koch and Tataru [7] established the global well-posedness of the classical Navier-Stokes equations with small initial data in the space BMO^{-1} . Then the question whether this result can be extended to

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the largest critical space $\dot{B}_{\infty,\infty}^{-1}$ had become of great interest among researchers, but it still remains open.

The first indication that such an extension might not be possible came in the work by Bourgain and Pavlović [1] who showed the norm inflation for the classical Navier-Stokes equations in $\dot{B}_{\infty,\infty}^{-1}$. More precisely, they constructed arbitrarily small initial data in $\dot{B}_{\infty,\infty}^{-1}$, such that mild solutions with this data become arbitrarily large in $\dot{B}_{\infty,\infty}^{-1}$ after an arbitrarily short time. Moreover, in [4] Cheskidov and Shvydkoy proved the existence of discontinuous Leray-Hopf solutions of the Navier-Stokes equations in $\dot{B}_{\infty,\infty}^{-1}$ with arbitrary small initial data.

Recently, Yu and Zhai [14] considered the fractional Navier-Stokes equations (1.1) with $\alpha \in (1/2, 1)$ and showed global well-posedness for small initial data in the largest critical space $\dot{B}_{\infty,\infty}^{1-2\alpha}$, conjecturing that the above mentioned ill-posedness results could not be extended to the hypodissipative case $\alpha < 1$.

Indeed, in the recent work [3] Cheskidov and Shvydkoy were able to prove the existence of discontinuous Leray-Hopf solutions in the largest critical space with arbitrarily small initial data for $\alpha \in [1, 5/4)$. However, the construction broke down for $\alpha < 1$.

In this paper we consider the case $\alpha \in [1, \infty)$ and demonstrate that the natural space for the norm inflation is not critical, but $\dot{B}_{\infty,\infty}^{-\alpha}$. Note that it is only critical in the classical case $\alpha = 1$, and it is not scaling invariant otherwise. More precisely, we prove the existence of a smooth solution with arbitrary small in $\dot{B}_{\infty,\infty}^{-\alpha}$ initial data that becomes arbitrary large in $\dot{B}_{\infty,\infty}^{-s}$ for all $s > 0$ in arbitrary small time. This recovers Bourgain and Pavlović's ill-posedness result in the case $\alpha = 1$, and shows the norm inflation in the largest critical space for $\alpha \in (1, 5/4)$, since $\dot{B}_{\infty,\infty}^{-\alpha} \subset \dot{B}_{\infty,\infty}^{1-2\alpha}$, $\alpha > 1$. Moreover, the case $\alpha > 1$ is particularly interesting since the norm inflation actually occurs in supercritical spaces that suggests that a small initial data result might be out of reach in those spaces. It is remarkable that the norm inflation holds even in the case $\alpha \geq 5/4$, where the global regularity is known. Hence the smooth solution that exhibits the norm inflation can actually be extended globally in time.

Our construction is similar to the one of Bourgain and Pavlović, but we have to deal with the lack of continuity of the bilinear operator corresponding to the fractional heat kernel on a modified Koch and Tataru type working space. This result is also based on the fact that a backwards energy cascade, harmless as far as the regularity of a solution is concerned, results in the growth of Besov norms with negative smoothness indexes. In this construction we also make sure that the initial data is space periodic and has a finite energy if considered on a torus. Namely, we show that

Theorem 1.1. *Let $\alpha \geq 1$. For any $\delta > 0$ there exists a smooth space-periodic solution $u(t)$ of (1.1) with the initial data*

$$\|u(0)\|_{\dot{B}_{\infty,\infty}^{-\alpha}} \lesssim \delta$$

that satisfies, for some $0 < T < \delta$ and all $s > 0$,

$$\|u(T)\|_{\dot{B}_{\infty,\infty}^{-s}} \gtrsim \frac{1}{\delta}.$$

Remark 1.2. We refer the reader to the beginning of section of Preliminaries for the definition of the symbol \lesssim .

We now recall some auxiliary concepts related to the plane waves, which are necessary in the sequel:

- The “diffusion” of a plane wave $v \cos(k \cdot x)$ in R^3 under the fractional Laplacian $-(-\Delta)^\alpha$ is given by

$$e^{-t(-\Delta)^\alpha} v \cos(k \cdot x) = e^{-|k|^{2\alpha} t} v \cos(k \cdot x)$$

Thus the magnitude of the diffusion of a plane wave dies down in time in the scale that is measured by $|k|^{2\alpha}$.

- It is easy to see that $u = e^{-|k|^{2\alpha} t} v \cos(k \cdot x)$ solves the system (1.1) when the wave vector k is orthogonal to the amplitude vector v .
- The nonlinear interaction of two such diffusions in the system (1.1) can be captured, and it only produces a slower diffusion if the two wave vectors are close.

We note that these observations, are the basis of the original argument of Bourgain and Pavlović in [1]. We will use them to construct a combination of such “diffusions” with least nonlinear interactions yet producing enough slower “diffusions” to cause the norm inflation in short time.

The rest of the paper is organized as: in Section 2 we introduce some notations that shall be used throughout the paper and some auxiliary results; in Section 3 we describe how the diffusions of plane waves interact in the fractional NSE system; in Section 4 we devote to proving Theorem 1.1.

2. PRELIMINARIES

2.1. Notation. We denote by $A \lesssim B$ an estimate of the form $A \leq CB$ with some constant C , and by $A \sim B$ an estimate of the form $C_1 B \leq A \leq C_2 B$ with some constants C_1, C_2 . For simplification of the notation, we denote $\|\cdot\|_p = \|\cdot\|_{L^p}$.

2.2. Semigroup operator $S_\alpha(t)$. Consider the Cauchy problem of the n dimensional dissipative equation with fractional power Laplacian,

$$(2.2) \quad \begin{cases} u_t + (-\Delta)^\alpha u = 0, \\ u(0) = \phi(x), \end{cases}$$

with $(x, t) \in \mathbb{R}^n \times [0, \infty)$.

Denote by \mathcal{F} and \mathcal{F}^{-1} the Fourier transform and inverse Fourier transform respectively. The solution of (2.2) can be written as

$$(2.3) \quad u(x, t) = S_\alpha(t) \phi(x) := \mathcal{F}^{-1} \left(e^{-t|\xi|^{2\alpha}} \mathcal{F}(\phi)(\xi) \right).$$

Thus $S_\alpha(t) := e^{-t(-\Delta)^\alpha}$ denotes the linear semigroup generated by the homogeneous linear fractional power dissipative equation (2.2).

Let \mathbb{P} denote the projection on divergence-free vector fields, which acts on a function ϕ as

$$\mathbb{P}(\phi) = \phi + \nabla \cdot (-\Delta)^{-1} \operatorname{div} \phi.$$

Lemma 2.1. [10] *There exists a constant $c > 0$ such that*

$$\|\nabla S_\alpha(t) \mathbb{P} \phi\|_\infty \leq c t^{-\frac{1}{2\alpha}} \|\phi\|_\infty$$

for any $t > 0$, and $\phi \in L^\infty$.

2.3. Norm of Besov spaces. For completeness we recall the equivalent norms for the homogeneous and inhomogeneous Besov spaces $\dot{B}_{\infty,\infty}^{-s}$ and $B_{\infty,\infty}^{-s}$ (c. f. [8]) respectively

$$(2.4) \quad \begin{aligned} \|f\|_{\dot{B}_{\infty,\infty}^{-s}} &= \sup_{t>0} t^{\frac{s}{2\alpha}} \|e^{-t(-\Delta)^\alpha} f\|_{L^\infty}, \\ \|f\|_{B_{\infty,\infty}^{-s}} &= \sup_{0<t<1} t^{\frac{s}{2\alpha}} \|e^{-t(-\Delta)^\alpha} f\|_{L^\infty}. \end{aligned}$$

Note that for the periodic space, for instance on \mathbb{T}^3 , the homogeneous and inhomogeneous spaces are equivalent (c. f. [11]). Thus,

$$\|f\|_{B_{\infty,\infty}^{-s}(\mathbb{T}^3)} = \|f\|_{\dot{B}_{\infty,\infty}^{-s}(\mathbb{T}^3)}.$$

It is then easy to observe that

$$(2.5) \quad \|f\|_{\dot{B}_{\infty,\infty}^{-s}(\mathbb{T}^3)} \leq \|f\|_{L^\infty(\mathbb{T}^3)},$$

since $\|e^{-t(-\Delta)^\alpha} f\|_{L^\infty} \leq \|f\|_{L^\infty}$.

2.4. Bilinear operator. Define the bilinear operator

$$(2.6) \quad \mathcal{B}_\alpha(u, v) = \int_0^t S_\alpha(t-\tau) \mathbb{P} \nabla \cdot (u \otimes v) d\tau = \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot (u \otimes v) d\tau.$$

As shown in [7, 9] the bilinear operator when $\alpha = 1$

$$\mathcal{B}(u, v) = \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u \otimes v) d\tau,$$

maps $X_T \times X_T$ into X_T continuously for a certain working space X_T . In [1], the continuity of \mathcal{B} on $X_T \times X_T$ plays an important rule to estimate the higher order iterations (the part y in the paper) of the nonlinear term. When $\alpha > 1$, it is difficult to choose such an appropriate working space and derive the continuity property. However we adopt the idea of Yoneda's work [13], and apply a relatively weaker estimate for the bilinear operator to control the nonlinear interactions. In the following, we show an estimate for the bilinear operator \mathcal{B}_α in L^∞ .

Lemma 2.2. *Let $u, v \in L^1(0, T; L^\infty)$ be such that $u \otimes v \in L^1(0, T; L^\infty)$. Then for all $\alpha > 0$, it satisfies*

$$(2.7) \quad \|\mathcal{B}_\alpha(u, v)\|_\infty \leq C \int_0^t \frac{1}{(t-\tau)^{1/2\alpha}} \|u(\tau)\|_\infty \|v(\tau)\|_\infty d\tau.$$

Proof: By the definition (2.6) and Lemma 2.1 we have

$$\begin{aligned} \|\mathcal{B}_\alpha(u, v)\|_\infty &\leq C \int_0^t \|S_\alpha(t-\tau) \mathbb{P} \nabla \cdot (u \otimes v)(\tau)\|_\infty d\tau \\ &\leq C \int_0^t \frac{1}{(t-\tau)^{1/2\alpha}} \|u(\tau)\|_\infty \|v(\tau)\|_\infty d\tau \end{aligned}$$

for any $u, v \in L^1(0, T; L^\infty)$. □

3. INTERACTIONS OF PLANE WAVES

3.1. The first iteration approximation of a mild solution. Let u be a solution to (1.1). We write it in the form

$$(3.8) \quad u = e^{-t(-\Delta)^\alpha} u_0 - u_1 + y$$

where

$$(3.9) \quad u_1(x, t) = \mathcal{B}_\alpha(e^{-t(-\Delta)^\alpha} u_0(x), e^{-t(-\Delta)^\alpha} u_0(x)).$$

A simple calculation shows that

$$(3.10) \quad y(t) = - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} [G_0(\tau) + G_1(\tau) + G_2(\tau)] d\tau,$$

where

$$(3.11) \quad \begin{aligned} G_0 &= \mathbb{P}[(e^{-t(-\Delta)^\alpha} u_0 \cdot \nabla) u_1 + (u_1 \cdot \nabla) e^{-t(-\Delta)^\alpha} u_0 + (u_1 \cdot \nabla) u_1] \\ G_1 &= \mathbb{P}[(e^{-t(-\Delta)^\alpha} u_0 \cdot \nabla) y + (u_1 \cdot \nabla) y + (y \cdot \nabla) e^{-t(-\Delta)^\alpha} u_0 + (y \cdot \nabla) u_1] \\ G_2 &= \mathbb{P}[(y \cdot \nabla) y]. \end{aligned}$$

Note that G_0 does not depend on y , G_1 is linear, and G_2 is quadratic in y .

In this section we show how the diffusions of plane waves interact in the fractional NSE system. These interactions are the basis for the constructions of initial data to produce the norm inflation.

3.2. Diffusion of a plane wave. As a first step, we consider the initial data being one single plane wave. Suppose $k \in \mathbb{R}^3$, $v \in \mathbb{S}^2$ and $k \cdot v = 0$. Let

$$u_0 = v \cos(k \cdot x).$$

Then $\nabla \cdot u_0 = 0$ and

$$(3.12) \quad e^{-t(-\Delta)^\alpha} v \cos(k \cdot x) = e^{-|k|^{2\alpha} t} v \cos(k \cdot x).$$

In fact the “diffusion” $e^{-t(-\Delta)^\alpha} v \cos(k \cdot x)$ of a plane wave solve (1.1) with vanishing pressure. And it is important to notice that for $s > 0$

$$\|v \cos(k \cdot x)\|_{\dot{B}_{\infty, \infty}^{-s}} \sim |k|^{-s}.$$

3.3. Interaction of plane waves. Now we consider the interaction of two different single plane waves. Suppose $k_i \in \mathbb{R}^3$, $v_i \in \mathbb{S}^2$ and $k_i \cdot v_i = 0$, for $i = 1, 2$. Let

$$u_1 = \cos(k_1 \cdot x) v_1, \quad u_2 = \cos(k_2 \cdot x) v_2.$$

To simplify our calculations we assume that $k_2 \cdot v_1 = \frac{1}{2}$. It then follows from a straightforward calculation that

$$\begin{aligned} & e^{-t(-\Delta)^\alpha} u_1 \cdot \nabla (e^{-t(-\Delta)^\alpha} u_2) \\ &= -e^{-(|k_1|^{2\alpha} + |k_2|^{2\alpha})t} v_2 \cos(k_1 \cdot x) \sin(k_2 \cdot x) (k_2 \cdot v_1) \\ &= -\frac{1}{4} e^{-(|k_1|^{2\alpha} + |k_2|^{2\alpha})t} v_1 (\sin((k_2 - k_1) \cdot x) + \sin((k_1 + k_2) \cdot x)). \end{aligned}$$

Hence

$$\begin{aligned} & \mathcal{B}_\alpha(e^{-t(-\Delta)^\alpha} u_1, e^{-t(-\Delta)^\alpha} u_2) \\ &= \frac{1}{4} v_1 \sin((k_2 - k_1) \cdot x) \int_0^t e^{-(|k_1|^{2\alpha} + |k_2|^{2\alpha})\tau} e^{-|k_2 - k_1|^{2\alpha}(t-\tau)} d\tau \\ &+ \frac{1}{4} v_1 \sin((k_1 + k_2) \cdot x) \int_0^t e^{-(|k_1|^{2\alpha} + |k_2|^{2\alpha})\tau} e^{-|k_1 + k_2|^{2\alpha}(t-\tau)} d\tau. \end{aligned}$$

Therefore, the interaction of the two plane waves is small in $\dot{B}_{\infty,\infty}^{-s}$ if neither the sum nor the difference of their wave vectors is small in magnitude. In the contrary, the interaction is sizable in $\dot{B}_{\infty,\infty}^{-s}$ if either the sum or the difference of their wave vectors is small in magnitude.

4. PROOF OF THEOREM 1.1

In this section we follow the idea from [1] to construct initial data which produces norm inflation for solutions to the fractional Navier-Stokes equation. From the discussions in Subsection 3.3 we see that the interaction of two plane waves is not enough to show the norm inflation, so need to use many waves. We also make sure that the initial data is space-periodic and smooth, which ensures the local existence of a smooth periodic solution to the fractional NSE. As we control its L^∞ norm, the solution will remain smooth until the time of the norm inflation.

4.1. Construction of initial data for the fractional NSE system. For a fixed small number $\delta > 0$ we will specify later the following initial data :

$$(4.13) \quad u_0 = r^{-\beta} \sum_{i=1}^r |k_i|^\alpha (v \cos(k_i \cdot x) + v' \cos(k'_i \cdot x))$$

with $\beta > 0$. We expect for each i , the interaction of the two plane waves $v \cos(k_i \cdot x)$ and $v' \cos(k'_i \cdot x)$ is sizable in $\dot{B}_{\infty,\infty}^{-s}$; while the interactions of plane waves of different i are small. Hence, by modifying the idea in [13], we choose

- Wave vectors: Let $\zeta = (1, 0, 0)$ and $\eta = (0, 0, 1)$. The wave vectors $k_i \in \mathbb{T}^3$ are parallel to ζ . The modulo $|k_0|$ will be taken to be a large integer, depending on r . The magnitude of k_i is defined by,

$$(4.14) \quad |k_i| = \left[2^{\frac{i}{\alpha}} |k_0| |k_{i-1}| \right], \quad i = 1, 2, 3, \dots, r.$$

The symbol $[a]$ denotes the least integer that is larger than or equal a . The wave vectors $k'_i \in \mathbb{T}^3$ are defined by

$$(4.15) \quad k'_i = k_i + \eta.$$

- Amplitude vectors: Let

$$(4.16) \quad v = (0, 0, 1), \quad v' = (0, 1, 0).$$

Hence

$$k_i \cdot v = k'_i \cdot v' = 0$$

which ensures the initial data is divergence free.

We first point out the following simple facts to further motivate the choices of the parameters.

Lemma 4.1. *Let $\gamma > 0$. With the choice (4.14)-(4.16), we have*

$$(4.17) \quad k_i \cdot v' = 0, \quad k'_i \cdot v = 1, \quad \forall \quad i = 1, 2, \dots, r.$$

$$(4.18) \quad \sum_{j < i} |k_j|^\alpha \sim |k_{i-1}|^\alpha \quad \text{and} \quad \sum_{j < i} |k'_j|^\alpha \sim |k'_{i-1}|^\alpha$$

$$(4.19) \quad \sum_{i=1}^r |k_i|^\gamma e^{-|k_i|^{2\alpha}t} \lesssim t^{-\frac{\gamma}{2\alpha}} \quad \text{and} \quad \sum_{i=1}^r |k'_i|^\gamma e^{-|k'_i|^{2\alpha}t} \lesssim t^{-\frac{\gamma}{2\alpha}}.$$

Proof: The first conclusion (4.17) is a simple fact due to the choice (4.14)-(4.16). By the definition (4.14), it is clear that $|k_{i-1}|^\alpha < \frac{1}{2}|k_i|^\alpha$, which easily implies the second statement (4.18).

By (4.14), we know that $|k_i|^\alpha \sim |k_i|^\alpha - |k_{i-1}|^\alpha$. Thus,

$$\sum_{i=1}^r |k_i|^\gamma e^{-|k_i|^{2\alpha}t} \sim \sum_{i=1}^r |k_i|^{\gamma-\alpha} (|k_i|^\alpha - |k_{i-1}|^\alpha) e^{-|k_i|^{2\alpha}t},$$

while the later one can be considered as a finite Riemman summation of the function $x^{\gamma/\alpha-1}e^{-x^2t}$. Therefore, for $\gamma > 0$ and $\alpha > 0$

$$\sum_{i=1}^r |k_i|^\gamma e^{-|k_i|^{2\alpha}t} \lesssim \int_0^\infty x^{\gamma/\alpha-1} e^{-x^2t} dx = t^{-\frac{\gamma}{2\alpha}} \int_0^\infty y^{\gamma/\alpha-1} e^{-y^2} dy \lesssim t^{-\frac{\gamma}{2\alpha}}.$$

It completes the proof of the lemma. □

Next we calculate the norms of the initial data.

Lemma 4.2. *For u_0 given in (4.13) and $\alpha > 0$ we have*

$$(4.20) \quad \|u_0\|_{\dot{B}_{\infty,\infty}^{-\alpha}} \lesssim r^{-\beta}.$$

Proof: Due to (3.12), we have that,

$$(4.21) \quad e^{-t(-\Delta)^\alpha} u_0 = r^{-\beta} \sum_{s=1}^r |k_s|^\alpha (v \cos(k_s \cdot x) e^{-|k_s|^{2\alpha}t} + v' \cos(k'_s \cdot x) e^{-|k'_s|^{2\alpha}t}).$$

Hence by Lemma 4.1,

$$\|u_0\|_{\dot{B}_{\infty,\infty}^{-\alpha}} \sim r^{-\beta} \sup_{0 < t < 1} t^{\frac{1}{2}} \sum_{s=1}^r |k_s|^\alpha \left(e^{-|k_s|^{2\alpha}t} + e^{-|k'_s|^{2\alpha}t} \right) \lesssim r^{-\beta}.$$

It completes the proof of the lemma. □

Lemma 4.3. *For u_0 given in (4.13) we have*

$$\|e^{-t(-\Delta)^\alpha} u_0\|_\infty \lesssim r^{-\beta} t^{-1/2}.$$

Proof: By (4.21) and Lemma 4.1, we infer that

$$\|e^{-t(-\Delta)^\alpha} u_0\|_\infty \lesssim r^{-\beta} \sum_{s=1}^r |k_s|^\alpha \left(e^{-|k_s|^{2\alpha}t} + e^{-|k'_s|^{2\alpha}t} \right) \lesssim r^{-\beta} t^{-1/2}.$$

□

4.2. Analysis of u_1 . As demonstrated in Subsection 3.1 we consider the decomposition

$$u = e^{-t(-\Delta)^\alpha} u_0 - u_1 + y.$$

Recall the definition (3.9)

$$u_1 = \mathcal{B}_\alpha(e^{-t(-\Delta)^\alpha} u_0, e^{-t(-\Delta)^\alpha} u_0).$$

By (4.16), (4.17), (4.21) and a straightforward calculation, it follows that

$$\begin{aligned}
 (4.22) \quad & (e^{-t(-\Delta)^\alpha} u_0 \cdot \nabla) e^{-t(-\Delta)^\alpha} u_0 \\
 &= -r^{-2\beta} \sum_{i=1}^r \sum_{j=1}^r |k_i|^\alpha |k_j|^\alpha e^{-(|k_i|^{2\alpha} + |k_j'|^{2\alpha})t} v' \cos(k_i \cdot x) \sin(k_j' \cdot x) \\
 &= -\frac{r^{-2\beta}}{2} \sum_{i=1}^r |k_i|^{2\alpha} e^{-(|k_i'|^{2\alpha} + |k_i|^{2\alpha})t} \sin(\eta \cdot x) v' \\
 &\quad - \frac{r^{-2\beta}}{2} \sum_{i \neq j}^r |k_i|^\alpha |k_j|^\alpha e^{-(|k_i|^{2\alpha} + |k_j'|^{2\alpha})t} \sin((k_j' - k_i) \cdot x) v' \\
 &\quad - \frac{r^{-2\beta}}{2} \sum_{i=1}^r \sum_{j=1}^r |k_i|^\alpha |k_j|^\alpha e^{-(|k_i|^{2\alpha} + |k_j'|^{2\alpha})t} \sin((k_j' + k_i) \cdot x) v' \\
 &\equiv E_0 + E_1 + E_2,
 \end{aligned}$$

where we used the fundamental formula $\cos x \sin y = [\sin(x+y) - \sin(x-y)]/2$.

Recall that $\eta \cdot v' = 0$, $(k_j' + k_i) \cdot v' = 0$ and $(k_j' - k_i) \cdot v' = 0$ for all i, j due to (4.17). Hence E_0 , E_1 and E_2 are divergence free vectors. Thus we can write

$$\begin{aligned}
 (4.23) \quad u_1 &= \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} E_0(\tau) d\tau + \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} E_1(\tau) d\tau \\
 &\quad + \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} E_2(\tau) d\tau \equiv u_{10} + u_{11} + u_{12}.
 \end{aligned}$$

We have the following estimates.

Lemma 4.4. *Let u_{10} be defined in (4.23) and $s > 0$. Then*

$$\begin{aligned}
 \|u_{10}(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-s}} &\gtrsim r^{1-2\beta}, \quad \text{for } |k_1|^{-2\alpha} \leq t \leq T, \\
 \|u_{10}(\cdot, t)\|_{\infty} &\lesssim r^{1-2\beta}, \quad \text{for all } t > 0.
 \end{aligned}$$

Proof: From (4.22) and (4.23), it follows by a straightforward calculation

$$\begin{aligned}
 u_{10} &= -\frac{r^{-2\beta}}{2} \int_0^t \sum_{i=1}^r |k_i|^{2\alpha} e^{-(|k_i'|^{2\alpha} + |k_i|^{2\alpha})\tau} e^{-|\eta|^{2\alpha}(t-\tau)} \sin(\eta \cdot x) v' d\tau \\
 &= -\frac{r^{-2\beta}}{2} \sin(\eta \cdot x) v' \sum_{i=1}^r |k_i|^{2\alpha} e^{-t} \frac{1 - e^{-(|k_i'|^{2\alpha} + |k_i|^{2\alpha} - 1)t}}{|k_i'|^{2\alpha} + |k_i|^{2\alpha} - 1} \\
 &\sim -\frac{r^{-2\beta}}{2} \sin(\eta \cdot x) v' \sum_{i=1}^r e^{-t} (1 - e^{-|k_i|^{2\alpha}t}).
 \end{aligned}$$

Hence, for $|k_1|^{-2\alpha} \leq t \leq T$ and $s > 0$

$$\|u_{10}(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-s}} \gtrsim r^{-2\beta} \cdot r \sup_{0 < t < 1} t^{\frac{s}{2\alpha}} e^{-|\eta|^{2\alpha}t} \gtrsim r^{1-2\beta}.$$

For all $t > 0$,

$$\|u_{10}(\cdot, t)\|_\infty \lesssim \frac{r^{-2\beta}}{2} \cdot r \lesssim r^{1-2\beta}.$$

It completes the proof of the lemma. \square

Lemma 4.5. *Let u_{11} and u_{12} be defined in (4.23). Then for $t > 0$*

$$\|u_{11}(\cdot, t)\|_\infty \lesssim r^{-2\beta}, \quad \|u_{12}(\cdot, t)\|_\infty \lesssim r^{-2\beta}.$$

Proof: By (4.22) and (4.23), it follows

$$\begin{aligned} u_{11} &= \frac{r^{-2\beta}}{2} \int_0^t \sum_{i \neq j}^r |k_i|^\alpha |k_j|^\alpha e^{-(|k_i|^{2\alpha} + |k_j'|^{2\alpha})\tau} e^{-|k_j' - k_i|^{2\alpha}(t-\tau)} \sin((k_j' - k_i) \cdot x) v' d\tau \\ &\sim \frac{r^{-2\beta}}{2} \sum_{i=1}^r \sum_{j < i}^r |k_i|^\alpha |k_j|^\alpha e^{-|k_i - k_j'|^{2\alpha}t} \frac{1 - e^{-(|k_i|^{2\alpha} + |k_j'|^{2\alpha} - |k_i - k_j'|^{2\alpha})t}}{|k_i|^{2\alpha} + |k_j'|^{2\alpha} - |k_i - k_j'|^{2\alpha}} \\ &\quad \cdot \sin((k_j' - k_i) \cdot x) v' \\ &\sim \frac{r^{-2\beta}}{2} \sum_{i=1}^r \sum_{j < i}^r |k_i|^\alpha |k_j|^\alpha t e^{-|k_i|^{2\alpha}t} \sin((k_j' - k_i) \cdot x) v' \end{aligned}$$

where we used the fact that $\frac{1-e^{-x}}{x}$ is bounded for $x > 0$. Hence, by (4.18) and (4.19) we infer that

$$\begin{aligned} \|u_{11}(\cdot, t)\|_\infty &\lesssim r^{-2\beta} \sum_{i=1}^r \sum_{j < i}^r |k_i|^\alpha |k_j|^\alpha t e^{-|k_i|^{2\alpha}t} \\ &\lesssim r^{-2\beta} \sum_{i=1}^r |k_i|^{2\alpha} t e^{-|k_i|^{2\alpha}t} \lesssim r^{-2\beta}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} u_{12} &= \frac{r^{-2\beta}}{2} \int_0^t \sum_{i=1}^r \sum_{j=1}^r |k_i|^\alpha |k_j|^\alpha e^{-(|k_i|^{2\alpha} + |k_j'|^{2\alpha})\tau} e^{-|k_i + k_j'|^{2\alpha}(t-\tau)} \sin((k_i + k_j') \cdot x) v' d\tau \\ &= \frac{r^{-2\beta}}{2} \sum_{i=1}^r \sum_{j=1}^r |k_i|^\alpha |k_j|^\alpha e^{-(|k_i|^{2\alpha} + |k_j'|^{2\alpha})t} \frac{1 - e^{-(|k_i + k_j'|^{2\alpha} - |k_i|^{2\alpha} - |k_j'|^{2\alpha})t}}{|k_i + k_j'|^{2\alpha} - |k_i|^{2\alpha} - |k_j'|^{2\alpha}} \\ &\quad \cdot \sin((k_i + k_j') \cdot x) v' \\ &\sim r^{-2\beta} \sum_{i=1}^r \sum_{j \leq i}^r |k_i|^\alpha |k_j|^\alpha e^{-(|k_i|^{2\alpha} + |k_j'|^{2\alpha})t} t \sin((k_i + k_j') \cdot x) v' \end{aligned}$$

Thus,

$$\begin{aligned} \|u_{12}(\cdot, t)\|_\infty &\lesssim r^{-2\beta} \sum_{i=1}^r \sum_{j \leq i}^r |k_i|^\alpha |k_j|^\alpha t e^{-|k_i|^{2\alpha}t} \\ &\lesssim r^{-2\beta} \sum_{i=1}^r |k_i|^{2\alpha} t e^{-|k_i|^{2\alpha}t} \lesssim r^{-2\beta}. \end{aligned}$$

It completes the proof of the lemma. \square

4.3. Analysis of y . In this section we analyze the part y of the solution. The idea is to control y using the estimate (2.7) of the bilinear operator \mathcal{B}_α in the space L^∞ .

Recall from Subsection 3.1 that

$$(4.24) \quad y(t) = - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} [G_0(\tau) + G_1(\tau) + G_2(\tau)] d\tau, \quad t \in [0, T].$$

Lemma 4.6. *With appropriate choice of β , r , T and $|k_1|$, we have, if $\alpha \geq 1$,*

$$\|y(t)\|_\infty \lesssim r^{1-3\beta} t^{\frac{1}{2}-\frac{1}{2\alpha}} + r^{2-4\beta} t^{1-\frac{1}{2\alpha}}$$

for $0 \leq t \leq T$.

Proof: It follows from (3.11) and (4.24) that

$$\begin{aligned} \|y(t)\|_\infty &\lesssim \|\mathcal{B}_\alpha(e^{-t(-\Delta)^\alpha} u_0, u_1(t))\|_\infty + \|\mathcal{B}_\alpha(u_1(t), u_1(t))\|_\infty \\ &\quad + \|\mathcal{B}_\alpha(e^{-t(-\Delta)^\alpha} u_0, y(t))\|_\infty + \|\mathcal{B}_\alpha(u_1(t), y(t))\|_\infty + \|\mathcal{B}_\alpha(y(t), y(t))\|_\infty. \end{aligned}$$

Applying the bilinear estimate (2.7), Lemmas 4.3, 4.4 and 4.5 we infer

$$\begin{aligned} \|\mathcal{B}_\alpha(e^{-t(-\Delta)^\alpha} u_0, u_1(t))\|_\infty &\lesssim \int_0^t \frac{1}{(t-\tau)^{1/2\alpha}} \|e^{-\tau(-\Delta)^\alpha} u_0\|_\infty \|u_1(\tau)\|_\infty d\tau \\ &\lesssim r^{1-3\beta} \int_0^t (t-\tau)^{-1/2\alpha} \tau^{-1/2} d\tau \\ &\lesssim r^{1-3\beta} t^{\frac{1}{2}-\frac{1}{2\alpha}}, \end{aligned}$$

where we used the boundedness of Beta function for $\alpha > 1/2$

$$\int_0^t (t-\tau)^{-1/2\alpha} \tau^{-1/2} d\tau = t^{\frac{1}{2}-\frac{1}{2\alpha}} B\left(\frac{1}{2}, 1 - \frac{1}{2\alpha}\right) \leq C t^{\frac{1}{2}-\frac{1}{2\alpha}}.$$

Similarly using the estimates obtained in previous two subsections, we have

$$\begin{aligned} \|\mathcal{B}_\alpha(u_1(t), u_1(t))\|_\infty &\lesssim \int_0^t \frac{1}{(t-\tau)^{1/2\alpha}} \|u_1(\tau)\|_\infty^2 d\tau \lesssim r^{2-4\beta} t^{1-\frac{1}{2\alpha}}, \\ \|\mathcal{B}_\alpha(e^{-t(-\Delta)^\alpha} u_0, y(t))\|_\infty &\lesssim \int_0^t \frac{1}{(t-\tau)^{1/2\alpha}} \|e^{-\tau(-\Delta)^\alpha} u_0\|_\infty \|y(\tau)\|_\infty d\tau \\ &\lesssim r^{-\beta} \int_0^t (t-\tau)^{-1/2\alpha} \tau^{-1/2} d\tau \sup_{0 < s < t} \|y(s)\|_\infty \\ &\lesssim r^{-\beta} t^{\frac{1}{2}-\frac{1}{2\alpha}} \sup_{0 < \tau < t} \|y(\tau)\|_\infty, \\ \|\mathcal{B}_\alpha(u_1(t), y(t))\|_\infty &\lesssim \int_0^t \frac{1}{(t-\tau)^{1/2\alpha}} \|u_1(\tau)\|_\infty \|y(\tau)\|_\infty d\tau \\ &\lesssim r^{1-2\beta} \int_0^t (t-\tau)^{-1/2\alpha} d\tau \sup_{0 < \tau < t} \|y(\tau)\|_\infty \\ &\lesssim r^{1-2\beta} t^{1-\frac{1}{2\alpha}} \sup_{0 < \tau < t} \|y(\tau)\|_\infty, \\ \|\mathcal{B}_\alpha(y(t), y(t))\|_\infty &\lesssim \int_0^t \frac{1}{(t-\tau)^{1/2\alpha}} \|y(\tau)\|_\infty^2 d\tau \lesssim t^{1-\frac{1}{2\alpha}} \left(\sup_{0 < \tau < t} \|y(\tau)\|_\infty \right)^2. \end{aligned}$$

Thus we have

$$\begin{aligned} \|y(t)\|_\infty &\lesssim r^{1-3\beta}t^{\frac{1}{2}-\frac{1}{2\alpha}} + r^{2-4\beta}t^{1-\frac{1}{2\alpha}} \\ &\quad + \left(r^{-\beta}t^{\frac{1}{2}-\frac{1}{2\alpha}} + r^{1-2\beta}t^{1-\frac{1}{2\alpha}} + t^{1-\frac{1}{2\alpha}} \sup_{0<\tau<t} \|y(\tau)\|_\infty \right) \sup_{0<\tau<t} \|y(\tau)\|_\infty. \end{aligned}$$

We choose large enough r , small enough $T > 0$ and appropriate $\beta, |k_1|$ such that,

$$(4.25) \quad r^{-\beta}t^{\frac{1}{2}-\frac{1}{2\alpha}} + r^{1-2\beta}t^{1-\frac{1}{2\alpha}} + t^{1-\frac{1}{2\alpha}}(r^{1-3\beta}t^{\frac{1}{2}-\frac{1}{2\alpha}} + r^{2-4\beta}t^{1-\frac{1}{2\alpha}}) \ll 1$$

for $0 \leq t \leq T$ (see a rigorous discussion on the choice of parameters in Subsection 4.4).

Thus we have the a priori bound by an absorbing argument

$$\|y(t)\|_\infty \lesssim r^{1-3\beta}t^{\frac{1}{2}-\frac{1}{2\alpha}} + r^{2-4\beta}t^{1-\frac{1}{2\alpha}}$$

for all $0 < t \leq T$. It proves the conclusion of the lemma. \square

4.4. Finishing the proof. Now we are ready to complete the proof of Theorem 1.1. Since u_0 is smooth and space-periodic, there exists $T^* > 0$ and a smooth space-periodic solution $u(t)$ to (1.1) on $[0, T^*)$ with $u(0) = u_0$, such that either $T^* = +\infty$ or

$$\limsup_{t \rightarrow T^*} \|u(t)\|_\infty = +\infty.$$

Lemmas 4.4, 4.5, and 4.6 imply that $T^* > T$. Now using (3.8), we combine the imbedding estimate (2.5), Lemmas 4.3, 4.4, 4.5 and 4.6 to obtain that, for $|k_1|^{-2\alpha} \leq t \leq T$

$$\begin{aligned} (4.26) \quad &\|u(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-s}} \\ &\geq \|u_{10}(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-s}} - \|u_{11}(\cdot, t)\|_\infty - \|u_{12}(\cdot, t)\|_\infty \\ &\quad - \|e^{-t(-\Delta)^\alpha} u_0\|_\infty - \|y(\cdot, t)\|_\infty \\ &\gtrsim r^{1-2\beta} \left(1 - r^{-1} - r^{\beta-1}t^{-\frac{1}{2}} - r^{-\beta}t^{\frac{1}{2}-\frac{1}{2\alpha}} - r^{1-2\beta}t^{1-\frac{1}{2\alpha}} \right) \\ &\gtrsim r^{1-2\beta} \left(1 - r^{\beta-1}|k_1|^\alpha - r^{-\beta}T^{\frac{1}{2}-\frac{1}{2\alpha}} - r^{1-2\beta}T^{1-\frac{1}{2\alpha}} \right). \end{aligned}$$

We first choose $\beta = 1/3$. Note that $\beta \in (0, \frac{1}{2})$. For any large integer r , let $T = r^{-\gamma}$ and $|k_1| = r^\zeta$ with positive γ, ζ which will be chosen such that for $\alpha \geq 1$,

$$\begin{cases} 0 < \zeta < \frac{1-\beta}{\alpha}, & \gamma > \frac{1-2\beta}{1-1/2\alpha}, \\ \gamma < 2\alpha\zeta. \end{cases}$$

To make the third inequality compatible with the first two inequalities, we need $1 - \beta < \alpha$ which is always true for $\beta \in (0, \frac{1}{2})$ and $\alpha \geq 1$.

For any large enough r , with the above choice of β, γ and ζ (hence T and $|k_1|$), we see that

$$(4.27) \quad r^{\beta-1}|k_1|^\alpha + r^{-\beta}T^{\frac{1}{2}-\frac{1}{2\alpha}} + r^{1-2\beta}T^{1-\frac{1}{2\alpha}} \ll 1, \quad \text{for } \alpha \geq 1.$$

It can be verified that both (4.25) and the assumption $|k_1|^{-2\alpha} < T$ in Lemma 4.4 are also satisfied for such β, γ and ζ . Given any $\delta > 0$ in Theorem 1.1, we then choose a suitable large r such that

$$r^{1-2\beta} \gtrsim \frac{1}{\delta}.$$

Therefore, it follows from (4.26) and (4.27)

$$\|u(\cdot, T)\|_{\dot{B}_{\infty, \infty}^{-s}} \gtrsim r^{1-2\beta} \gtrsim \frac{1}{\delta}.$$

On the other hand, by Lemma 4.2 and the specific $\beta = \frac{1}{3} \in (0, \frac{1}{2})$, the initial data u_0 satisfies

$$\|u_0\|_{\dot{B}_{\infty,\infty}^{-\alpha}} \lesssim r^{-\beta} \lesssim \delta.$$

Thus we proved Theorem 1.1.

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DEPARTMENT OF MATHEMATICS, STAT. AND COMP.SCI., UNIVERSITY OF ILLINOIS CHICAGO, CHICAGO, IL 60607, USA

E-mail address: acheskid@math.uic.edu

DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF COLORADO BOULDER, BOULDER, CO 80303, USA

E-mail address: mimi.dai@colorado.edu